sciences at Paris. The curve and its graph are given without name in Cramer's *Introduction à l'analyse des lignes courbes*, 1750, pp. 188, 478, and in O. Gherli's *Elementi teorico-pratici delle matematiche pure*, Vol. IV, Modena, 1773, pp. 343, 345.

The question still remains to be solved, who first used the name "Rolle's curve" for $xy^2 = a(y + x)^2$ and what connection, if any, Rolle had with it.[1] Who can throw further light on this?

THE MATHEMATICS OF AËRODYNAMICS.[2]

By EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

1. In the development of a branch of engineering there are several stages, sometimes long drawn out and poorly differentiated, sometimes brief and sharply defined.

There is the stage in which the bold inventors, trying one way and another, often with personal danger or even disaster, accomplish the first crude beginnings. In ship construction this epoch is prehistoric; in steam engineering it starts, at least in story, with Watts; in electrical engineering it lies within the memory of men still living; in aëronautics it stretches over a long period admirably described by A. F. Zahm in his book *Aërial Navigation*. The first essays in aëronautics, if we overlook the legendary Icarus, were the balloon and the balloon-type machine; progress with the heavier-than-air type may well be dated from Langley and the Wrights, though experiments with gliders are known for some time previous to them.

There is another stage—that in which the properly equipped mathematician and physicist construct a theory to correspond to the branch of engineering and by the indications of that theory aid materially in hastening the development of improved types of machines, or at least, if they arrive too late to help in the major development, they assist in codifying the fundamental scientific principles which underlie the subject and make possible systematic training of the youth to carry on the advance in details. The development of mathematics and physics was sufficient before the advent of electrical engineering to enable major advances to be made by William Thomson (later Lord Kelvin), by Oliver Heaviside, and by M. I. Pupin, to mention only three. In the case of the simpler parts of mechanical engineering and of naval architecture, the pressing necessities of prehistoric and early historic man were such that the art far outstripped the

[1] As far as I now know, the only curve which prior to 1896 may be said to have been associated with the name of Rolle is mentioned by J. Bernoulli in a letter to Leibniz, dated May 7, 1701. Bernoulli there refers to the "ultima Rolii curva a Varignonio delineata

$$y = 2 + \sqrt{(4 + 2x)} + \sqrt{(4x)},"$$

Got. Gul. Leibnitii et Johan. Bernoullii Commercium Philosophicum et Mathematicum, Tomus secundus, Lausannæ & Genevæ, MDCCXLV, pp. 41, 42, 44.

[2] Address read before the Mathematical Association of America at its annual meeting, at Columbia University, New York, Saturday, December 30, 1916.

science; indeed the limitations of our mathematical and physical knowledge of the behavior of liquids are still so serious that when we wish a cup-defender we have to rely almost entirely upon the fine sense of a genius like Herreshoff instead of upon the theories and calculations of engineers like Professor C. H. Peabody and Rear-Admiral Taylor.

In the development of the airplane, however, since the early experimental and inventive successes of the Wrights and their competitors and followers in France, much has been accomplished in a single decade by the mathematician and physicist, in particular by G. H. Bryan, whose book *Stability in Aviation* was epochal, and by L. Bairstow, of the National Physical Laboratory, Teddington, England, whose experimental and theoretical work has done so much toward rendering Bryan's ideas directly and surely applicable to actual flying machines. I have been fortunate enough under commission from our government to add my mite to the theoretical advance (*Motion of an Airplane in Gusts*), and we in this country should all do much homage to the pioneer work which Lieut. J. C. Hunsaker, U. S. N., performed at the Massachusetts Institute of Technology.

There is a third stage, a relatively late stage in this uprearing of a branch of engineering. Here we find the scientific and technical knowledge so codified in text-books and manuals that it is not particularly hard for a school of technology, properly equipped with laboratories and teaching staff, to take in as raw material average young fellows and turn out as manufactured product tolerably competent engineers, able to design and make minor improvements in standard machines. This stage is not yet quite attained in aëronautics, but with the obviously great technical advances which have come about with such perplexing rapidity during the past two or three years under the pressure of war necessities in Europe and in this country, and with the presumably great theoretical advances which have accompanied them, it certainly will not take long after the war shall have terminated and after the present belligerent governments shall have ventured to divulge the ways and means and ends in aëronautics which they now guard so sedulously—it certainly will not then take long to work down the science and art of aëronautical engineering to the point where it takes its place in this third stage with mechanical, civil, and electrical engineering. Already a preliminary codification may be found in the course prepared by Messrs. Klemin and Huff, on the basis of the course given by their former teacher, Dr. Hunsaker, which is now appearing serially in the recently established, interesting and important journal *Aviation and Aëronautical Engineering*.

My subject as announced by your program committee is the "Mathematics of Aërodynamics," and if I have been long in my preamble it is merely because I feel that before we examine one special phase of aëronautics, the phase that interests us here most nearly, we should first take a broad view of the whole field to acquire the perspective necessary for a just appraisal of the value and of the limitations of mathematics in engineering. Mathematicians as investigators are concerned chiefly with but a single stage of the work. They cannot aid much in the first hardy adventures; they cannot help greatly in the final codifica-

tion—except as they strive as teachers better to prepare the freshman and the sophomore in his mathematical courses for the technical work to which he looks forward; they can help materially in the intermediate stage, provided they have schooled themselves in that style of mathematics which is suited to the treatment of engineering problems, that is in mechanics and mathematical physics.

2. In discussing aëronautical machines mathematically, or otherwise, a sharp distinction must be made between the lighter-than-air and the heavier-than-air types—the balloon and the plane types, as we may briefly designate them.

For the balloon type the first mathematical work will naturally be very elementary, dealing with buoyancy, with the expansion and contraction of gases owing to rise and fall in the level of flight and to changes of temperature. Arithmetic and algebra alone or the simplest portions of the calculus will suffice. If the coefficients of resistance to motion are known or assumed for different shapes of balloon, simple calculations will determine the power necessary for uniform rectilinear motion. As resistances are supposed to vary with some power, generally the square, of the velocity, easy problems in the calculus may be made regarding stopping and starting with no further use of differential equations than the "variables separable" type.

In discussing the airplane some small use of trigonometry is necessary. The pressure P in a plane surface of area S, which is taken as a first approximation to a wing, is assumed to be normal to the plane (the viscous tangential drag being neglected), and of the form $P = kSv^2 \sin i$, where i is the angle between the wing and the direction of motion. This is resolved vertically and horizontally to give the "lift" L and "drift" D as

$$L = kSv^2 \sin i \cos i, \qquad D = kSv^2 \sin^2 i.$$

In uniform horizontal flight $L = W$, the weight of the machine, and $D = F$, the thrust of the propeller. (To a first approximation the resistance of all surfaces other than the main wing of the monoplane or the pair of wings of the biplane are neglected.) The power necessary to drive the wing through the air is then nearly

$$\frac{Dv}{550} = \frac{W^2}{550kSv}.$$

This shows that for a given machine, the power is inversely as the velocity. The apparent anomaly is explicable by virtue of the fact that at higher velocities the machine flies at smaller angles of incidence i. A pretty exercise may be had by assuming that the resistance other than that due to the wings may be lumped together in a term Cv^2 added to the drift D so that the power becomes

$$\frac{W^2}{550kSv} + \frac{Cv^3}{550};$$

the minimum power may then be determined with its corresponding speed; and the further result that for the speed which minimizes the power, the weight per horsepower varies inversely as the speed.

There are a number of such simple applications of trigonometry and calculus which may be introduced in many a course in mathematics for the purpose of enlivening the work. One particularly interesting calculation is found by contrasting the expression for the power which has been obtained from the assumption (known to Euler) that the pressure varies with the sine of i with the expression which would be obtained from the erroneous assumption (generally attributed to Newton) that the pressure varies with the square of the sine of i. Those of you who are interested in the application of elementary mathematics to the airplane will find a great deal of suggestion in Painlevé and Borel's *L'Aviation*.

I wish next to mention those parts of advanced mathematics which are used by writers on aërodynamics. Beginning with Kirchhoff and Lord Rayleigh and reaching its culmination in recent work by Sir George Greenhill, the theory of functions of a complex variable has been indispensable in deriving theoretical formulas for the pressure P of a stream of air on a wing. Problems in hydrodynamics which take account of eddy (or vortical) motion or of internal friction (viscosity) are so difficult to solve that one may with reasonable accuracy say that for theoretical work in aërodynamics, eddies and friction must be disregarded; and for similar reasons the fluid motion is ordinarily restricted to plane motion. There are to be sure "end-effects" when a wing moves through the air, but owing to the length of span an approximation of value may be had by neglecting the "end-effects" and assuming that the motion of the air in all planes perpendicular to the edge of the wing is identical. Now by the method of "conformal representation" of the theory of functions the pressure exerted on a plane wing of various shapes, by the motion of the air, may in some cases be calculated, and the center of pressure may also be found.

These theoretical determinations of pressure and center of pressure are not precisely verified when experimentally measured in the wind tunnel and it is now customary to use the experimental values in place of those theoretically calculated. Nevertheless no student of theoretical aërodynamics can afford to be ignorant of the elements of hydrodynamics, including the theory of the reactions of streams on given contours, and this means that he must have some knowledge of the theory of functions of a complex variable, or rather let me say, some knowledge of the means of applying the principle of conformal mapping to the setting up of the functions which gives the map of one boundary and enclosed region upon another boundary and enclosed region. It is unhappily true that many extended and beautiful courses on the theory of functions leave the student wholly unable to carry through the applications necessary to the solution of these practical, or at least semi-practical, problems in hydromechanics. Might I suggest to all who teach the complex variable and the conformal map the possibility of drawing on uniplanar fluid motion as a source of instructive exercises?

3. Owing partly to the divergence between theory and experiment in fluid motion in simple cases and partly to the impossibility of obtaining a theoretical hydrodynamic solution for the motion of so complicated an object as an airship

or airplane, it is necessary in the main to treat the ship or plane as a rigid body to which are applied certain forces (resultant fluid pressures and moments of pressures) determined by wind tunnel experiments upon models. The problem of the motion of the machine then becomes one in the dynamics of a rigid body free to move in space (*i. e.*, with six degrees of freedom) subject to known forces. It is customary to refer the motion to axes fixed in the machine and moving with it. Now unfortunately moving axes are regarded as belonging to really advanced rigid dynamics. Routh in his two volumes *Elementary and Advanced Rigid Mechanics* leaves moving axes to the beginning of the Advanced Part. Single volume treatises such as Loney's *Dynamics of a Particle and of Rigid Bodies* are apt to stop before reaching moving axes. It is not at all unlikely that the present interest in the dynamics of flight will force us so to alter our texts and our courses as to include the theory of moving axes; for we cannot expect students to prolong their studies of mechanics unduly before entering upon the motion of aërial machines.

Whether referred to moving or fixed axes the equations of motion of an airplane or airship, with the forces represented by empirical equations experimentally determined, are too difficult to integrate without approximation. The first case treated is that of motion in a line nearly straight and inclined to the horizontal at a nearly constant angle. By considering the motion as made up of small departures from uniform flight in straight lines, the differential equations of the motion become linear equations with constant coefficients, and their solution is therefore a matter of well-known, though sometimes very tedious, routine. And right here allow me to intercalate the remark that we must not ourselves disdain arithmetic nor allow our students to be discouraged by it; there is current all too much of a feeling that only the beautiful general theories of mathematics are worthy the attention of real mathematicians and their students; the art and the science of getting a sufficiently accurate numerical solution of a numerical problem are of equal importance and dignity with general theory in all applications of mathematics.

The utility of the solution of the nearly uniform motion of the machine is found in the discussion of dynamical stability. The approximate equations are solved, as all linear equations with constant coefficients are solved, in terms of exponential functions with or without combination with trigonometric terms. If a machine dynamically stable is slightly disturbed from uniform motion the small departures from that standard state will die out as time goes on, *i. e.*, the exponential functions must have negative exponents like e^{-at}. It is a pretty and none too easy algebraic problem, solved in Routh's *Rigid Dynamics* but not in many books on algebra, to determine the conditions under which an algebraic equation has its real roots, and the real parts of its imaginary roots, negative. The practical importance of dynamical stability in a machine is that under normal atmospheric conditions and for short periods of time the machine may safely be flown "hands off" and the pilot has therefore some freedom to attend to other matters than the guiding of his craft.

I have been told that the theory of Bryan, recast by Bairstow, and supplemented by data of wind tunnel experiments, has enabled a machine to be designed and constructed which actually has been flown "hands off." If this be true it marks a great triumph for mathematical physics; whether it is true or not, we can confidently assert that to the English training in mathematical physics is in no small degree due the great and sudden advance in airplane design and the great success in aircraft warfare which have been realized in England in relatively few months.

4. I have shown you that the mathematics of aërodynamics leads from elementary algebra and arithmetic to the theory of functions of a complex variable and to the solution of linear differential equations with constant coefficients. I have shown how the theoretical work has come in that stage of the development of aëronautical engineering where it has been of real help in rapidly advancing the art. There are always with us branches of engineering and physics in which the right kind of mathematics is of great value for the rapid advance of those branches. This right kind of mathematics is the good old traditional Cambridge, England, type, the mathematics of Newton, of Green, of Maxwell, of Kelvin, of Rayleigh, a type of mathematics which in this country, owing perhaps to our preponderance of German-trained mathematicians, has all too little prestige. I sometimes wonder whether we do right to aim so exclusively at the continental type. It is worthy of note that our two great native mathematicians, George W. Hill and J. Willard Gibbs, were concerned with the applied side. May it not be that we in this country have such a natural bent toward the practical that a diligent cultivation of the British sort of mathematics would find a readier response among our students?

We are here not as research mathematicians but as teachers of collegiate mathematics. Our country has great industrial problems of peace and war to solve, and every one of us must help as he may. As we bend the mathematical twig, so will the tree incline. Let us without prejudice consider our curricula and with open mind introduce any necessary changes to make sure that the type of mathematics which we place before our students is that which will contribute most to the victory of our country in time of stress and to her prosperity in times of peace.

SPRING MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section of the Mathematical Association of America was held at Carleton College on May 4, 1918. Seventy-eight people attended, including Edla G. Berger of College of St. Catherine, Father W. E. Etzel of St. Thomas College, C. H. Gingrich of Carleton College, chairman of the section, Jessie G. Quigley of College of St. Teresa, G. N. Bauer, W. H. Bussey, R. R. Shumway, H. L. Slobin, R. M. Barton of the University of Minnesota, members of the Association. All in attendance at the meeting were, as the guests of Carleton College, most hospitably entertained through the day and at luncheon.